Answers Exam Introduction to Logic (CS&MA)

1\. a. Translation key:
Z: We're going to that zoo.
A: There are spiders.
S: There are snakes.

Translation: $Z \to \neg(A \vee S)$

b. Translation key:
S: You can feed the sharks.
D: They die.
F: You feed the sharks.

Translation: $S \wedge (\neg F \to \neg D)$

2\. a. Translation key
j: John
$K(x,y)$: x knows y
$M(x)$: x is a mathematician
$L(x,y)$: x likes y
$C(x)$: x is a computer scientist

Translation: $\forall x((K(j,x) \wedge M(x)) \to \exists y(C(y) \wedge L(x,y)))$

b. Translation key
$C(x)$: x is a computer scientist.
$K(x,y)$: x knows y
$P(x)$: x is a philosopher.
$M(x,y)$: x has met y at some time.
$M(x)$: x is a mathematician

Translation: $\forall x(C(x) \to \exists y(P(y) \wedge K(x,y)) \wedge \neg\exists z(M(z) \wedge M(y,z)))$

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3 a.

1.	$\forall x(\exists y \neg R(x,y) \to P(x))$	
2.	$\forall x \forall y (x=y \to \neg R(x,y))$	
3.	$\exists y \neg R(a,y) \to P(a)$	$\forall$ Elim (1)
4.	$\forall y (a=y \to \neg R(a,y))$	$\forall$ Elim (2)
5.	$a=a \to \neg R(a,a)$	$\forall$ Elim (4)
6.	$a=a$	$=$ Intro
7.	$\neg R(a,a)$	$\to$ Elim (5, 6)
8.	$\exists y \neg R(a,y)$	$\exists$ Intro (7)
9.	$P(a)$	$\to$ Elim (3, 8)

b.

1.	$\forall x \forall y \forall z ((R(x,y) \land R(x,z)) \to R(y,z))$	
2.	$\forall x R(x,x)$	
3.	$\boxed{a}$	
4.	$\boxed{b}$	
5.	$R(a,b)$	
6.	$\forall y \forall z ((R(a,y) \land R(a,z)) \to R(y,z))$	$\forall$ Elim (1)
7.	$\forall z ((R(a,b) \land R(a,z)) \to R(b,z))$	$\forall$ Elim (6)
8.	$(R(a,b) \land R(a,a)) \to R(b,a)$	$\forall$ Elim (7)
9.	$R(a,a)$	$\forall$ Elim (2)
10.	$R(a,b) \land R(a,a)$	$\land$ Intro (5, 9)
11.	$R(b,a)$	$\to$ Elim (8, 10)
12.	$R(a,b) \to R(b,a)$	$\to$ Intro (5–11)
13.	$\forall y (R(a,y) \to R(y,a))$	$\forall$ Intro (4–12)
14.	$\forall x \forall y (R(x,y) \to R(y,x))$	$\forall$ Intro (3–13)

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4\. a

A	B	C	$A \leftrightarrow$	$(\neg B$	$\rightarrow$	$\neg C)$	$(A \wedge$	$(C \rightarrow B))$	$\vee$	$(\neg A$	$\wedge C$	$\wedge$	$\neg B)$
T	T	T	T	F	T	F	T	T	T	F	F	F	F
T	T	F	T	F	T	T	T	T	T	F	F	F	F
T	F	T	F	T	F	F	F	F	F	F	F	F	T
T	F	F	T	T	T	T	T	T	T	F	F	F	T
F	T	T	F	F	T	F	F	T	F	T	T	F	F
F	T	F	F	F	T	T	F	T	F	T	F	F	F
F	F	T	T	T	F	F	F	F	T	T	T	T	T
F	F	F	F	T	T	T	F	T	F	T	F	F	T

In each row both formulas get the same truth value.
Therefore the formulas are tautologically equivalent.

b.

A	B	$((A \rightarrow B)$	$\vee$	$(A \leftrightarrow B))$	$\wedge$	$(\neg A$	$\vee B)$
T	T	T	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	T	T	F	T	T	T
F	F	T	T	T	T	T	T

In the second row the formula gets truth value F.
Therefore it is not a tautology.

4 c.	$a=b$	$Tet(a)$	$Cube(b)$	$a=b \land (Tet(a) \lor Cube(b))$		$Tet(a) \leftrightarrow \neg Cube(b)$		$a=b \to (Tet(a) \lor \neg Cube(b))$		
*	T	T	T	T	T	F	F	T	T	F
	T	T	F	T	T	T	T	T	T	T
	T	F	T	T	T	T	F	F	F	F
	T	F	F	F	F	F	T	T	T	T
	F	T	T	F	T	F	F	T	T	F
	F	T	F	F	T	T	T	T	T	T
	F	F	T	F	T	T	F	T	F	F
	F	F	F	F	F	F	T	T	T	T

The row marked with * is spurious. In the third row, the premises are true, but the conclusion is false. Therefore, the conclusion is not a logical consequence of the premises.

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3. a. $\exists x \forall y(\text{Cube}(y) \rightarrow x=y)$ or $\exists x \forall y(\text{Cube}(y) \leftrightarrow x=y)$

b. (i) F
(ii) T
(iii) F
(iv) T
(v) T
(vi) F
(vii) T

c. One can remove object d or e, f.

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6. $\neg(A \vee (B \wedge C)) \vee (A \wedge B)$

$\Leftrightarrow$

$(\neg A \wedge \neg(B \wedge C)) \vee (A \wedge B)$

$\Leftrightarrow$

$(\neg A \wedge (\neg B \vee \neg C)) \vee (A \wedge B)$

$\Leftrightarrow$

$(\neg A \vee (A \wedge B)) \wedge ((\neg B \vee \neg C) \vee (A \wedge B))$

$\Leftrightarrow$

$(\neg A \vee A) \wedge (\neg A \vee B) \wedge ((\neg B \vee \neg C) \vee (A \wedge B))$

$\Leftrightarrow$

$(\neg A \vee A) \wedge (\neg A \vee B) \wedge (A \vee \neg B \vee \neg C) \wedge (B \vee \neg B \vee \neg C)$

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7 a. $\exists x \forall y\, R(x,y) \to \forall x \exists y\, Q(y,x)$

$\Leftrightarrow$

$\exists x \forall y\, R(x,y) \to \forall z \exists y\, Q(y,z)$

$\Leftrightarrow$

$\exists x \forall y\, R(x,y) \to \forall z \exists w\, Q(w,z)$

$\Leftrightarrow$

$\forall x (\forall y\, R(x,y) \to \forall z \exists w\, Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y (R(x,y) \to \forall z \exists w\, Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y \forall z (R(x,y) \to \exists w\, Q(w,z))$

$\Leftrightarrow$

$\forall x \exists y \forall z \exists w (R(x,y) \to Q(w,z))$

$\cancel{\Leftrightarrow}$ $\Leftarrow$

$\forall x \forall z \exists w (R(x,f(x)) \to Q(w,z))$

$\Leftarrow$

$\forall x \forall z (R(x,f(x)) \to Q(g(x,z),z))$

b.

A B C D	$(A \vee \neg B) \wedge (\neg A \vee B) \wedge (\neg C \vee \neg D \vee B) \wedge A \wedge (\neg A \vee C)$
T T T F	T T F T T F T T T T F T T T F T T T T T T F T T T

The formula is satisfiable.

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8a. $\neg\forall x(Loves(x, mostlike(x)) \to Loves(x,x))$

b. $\forall x(Loves(x, mostlike(a)) \vee x=b)$

c. $\exists x(Loves(x,a) \wedge \forall y(y = mostlike(a) \to Loves(x,y))) \vee \neg Loves(b,a)$

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9 a. $1 \notin \{2,3\}$, so $M(a) \notin M(Q)$ or $[\![a]\!]^M_h \notin M(Q)$. Therefore $M \not\models Q(a)\,[h]$.
Therefore $M \not\models Q(a) \to (Q(b) \vee R(x,y))\,[h]$.

b. i $3 \in \{2,3\}$, so $h[x/3](x) \in M(Q)$ or $[\![x]\!]^M_{h[x/3]} \in M(Q)$. Therefore $M \models Q(x)\,[h[x/3]]$.
ii $\langle 2,3\rangle \notin \{\langle 1,3\rangle, \langle 2,2\rangle, \langle 2,1\rangle, \langle 3,3\rangle\}$, so $\langle h[x/3](z), h[x/3](x)\rangle \notin M(R)$
or $\langle [\![z]\!]^M_{h[x/3]}, [\![x]\!]^M_{h[x/3]}\rangle \notin M(R)$. Therefore $M \not\models R(z,x)\,[h[x/3]]$.
From i and ii it follows that $M \not\models Q(x) \to R(z,x)\,[h[x/3]]$.
Therefore $M \not\models \forall x\,(Q(x) \to R(z,x))\,[h]$.

c. $1 \in \{1,3\}$, so $h[x/1][y/1](x) \in M(P)$ or $[\![x]\!]^M_{h[x/1][y/1]} \in M(P)$.
Therefore $M \models P(x)\,[h[x/1][y/1]]$, so $M \not\models \neg P(x)\,[h[x/1][y/1]]$.
Therefore $M \models \neg P(x) \to (Q(x,y) \wedge R(y,y))\,[h[x/1][y/1]]$.
So $M \models \exists y\,(\neg P(x) \to (Q(x,y) \wedge R(y,y)))\,[h[x/1]]$.
Therefore $M \models \exists x\,\exists y\,(\neg P(x) \to (Q(x,y) \wedge R(y,y)))\,[h]$.